Unified Octonionic Representation of the 10-13 Dimensional Clifford Algebra

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Abstract

We give a one dimensional octonionic representation of the different Clifford algebra $Cliff(5,5) \sim Cliff(9,1)$, $Cliff(6,6) \sim Cliff(10,2)$ and lastly $Cliff(7,6) \sim Cliff(10,3)$ which can be given by 8×8 real matrices taking into account some suitable manipulation rules.

I. INTRODUCTION

Since a long time, it has been conjectured that there exists a possible connection between the different members of the ring division algebra $(\mathcal{R}, \mathcal{C}, \mathcal{H}, \mathcal{O})$ and the critical dimensions of the Green-Schwarz superstring action [1–3]. Especially, the octonionic case has gained much attention due to its possible relation to the 10 dimensions physics [4–16]. Not just strings,

but even extended to p-branes, octonions are usually related to different 10 , 11 dimensions p-branes [17] and we would expect that the new M, F, S theories to be no exception.

The 10 dimensions superstring (Green-Schwarz) action is defined in terms of 16 components Majorana-Weyl spinor. The κ symmetry removes half of this degrees of freedom and renders the theory dependant of just 8 real components. So, trying to write a lagrangian for superstrings, we meet one of its many miracles, the theory knows how to do the D=10 Lorentz group with only 8×8 matrices but we don't. In this article, we would like to show the possibility of writing D=10 Clifford Algebra using one single octonions. To this end, we find a surprise the formulation allows the minimal, not just D=10 but, D=13 Cliff(7,6).

The idea is to know how to translate some real $n \times n$ ($\mathcal{R}(n)$) matrices to their corresponding complex and quaternionic matrices [18], in general, which can be extended to the octonionic algebra. It is well known from a topological point of view that any $\mathcal{R}^{2n}$ is trivially a $\mathcal{C}^n$ complex manifold and any $\mathcal{R}^{4n}$ is also a trivial quaternionic manifold $\mathcal{H}^n$, whereas, any $\mathcal{R}^{8n}$ is again a trivial $\mathcal{O}^n$ octonionic manifold. And, as any $\mathcal{R}^n$ is isomorphic as a vector space to $\mathcal{R}(n)$ matrices, we would expect

$$\mathcal{R}(2n) \rightarrow \mathcal{C}(n); \tag{1}$$

$$\mathcal{R}(4n) \rightarrow \mathcal{H}(n); \tag{2}$$

$$\mathcal{R}(8n) \rightarrow \mathcal{O}(n). \tag{3}$$

The aim of the paper is to illustrate how this structural isomorphism[1] works and to construct different Clifford algebra over quaternions and octonions. The paper is organized as follows: In section II. , just for completion, we prove this isomorphism for the complex numbers. In section III. we meet the new problems due to the non-commutativity of quaternions. We construct the quaternionic case in full details which will be a warm up as well as a good

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[1]Actually the isomorphism does not hold for the octonionic case as it is evident that matrix algebra is associative whereas octonions are not. Nevertheless, we can find some translation rules between $\mathcal{O}$ and R^8.

guide for the octonionic case which we treated at [19] and will be summariezed at section IV. emphasising the main differences and similarities in confront the quaternionic case. Then in section V. we show how to construct Lie Algebra (classical) from octonions. Adding one rule to overcome the non-associativity, we will be able to write down a Cliff(7,6) that can be represented by two sets of 8×8 real matrices defined with some simple manipulation rules. Lastly in section VI. we present our conclusions as well as some further open points. The construction is physically, not mathematically, motivated since, without the strange κ symmetry, no one would try to look for this formulation.

II. COMPLEX NUMBERS

To prove (1) , the idea goes as follows : For complex variables, one can represent any complex number z as an element of $\mathcal{R}^2$

$$z = z_0 + z_1 e_1, \equiv Z = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}. \tag{4}$$

The action of 1 and e_1 induce the following matrix transformations on Z ,

$$1.z = z.1 = z \equiv Z = \mathbb{1}Z, \tag{5}$$

while

$$e_1.z = z.e_1 = z_0 e_1 - z_1 \tag{6}$$

$$\equiv E_1 Z \tag{7}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} -z_1 \\ z_0 \end{pmatrix}. \tag{8}$$

Now, we have a problem, these two matrices $\mathbb{1}$ and E_1 are not enough to form a basis for $R(2)$. The solution of our dilemma is easy. We should also take into account

$$z^* = z_0 - z_1 e_1 \equiv Z^* = \begin{pmatrix} z_0 \\ -z_1 \end{pmatrix} \tag{9}$$

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so, we find

$$1^*.z = z^* \tag{10}$$

$$\equiv \mathbb{1}^* Z = Z^* \tag{11}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} z_0 \\ -z_1 \end{pmatrix}, \tag{12}$$

and

$$e_1.z^* = z^*.e_1 = z_0 e_1 + z_1 = e_1^* z \tag{13}$$

$$\equiv E_1 Z^* = E_1^* Z \tag{14}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_0 \\ -z_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_0 \end{pmatrix}. \tag{15}$$

Having, these four matrices $\{\mathbb{1}, \mathbb{1}^*, E_1, E_1^*\}$, (1) is proved. Lastly, we would like to mention that the relation between $\{\mathbb{1}^*, E_1, E_1^*\}$ and the quaternionic imaginary units , defined in the next section is the exact reason for the possible formulation of the 2 dimensions geometry in terms of quaternions [20].

III. QUATERNIONS

For quaternions, being non commutative, one should differentiate between right and left multiplication, (our quaternionic algebra is given by $e_i .e_j = -\delta_{ij} + \epsilon_{ijk}e_k$, and $i, j, k = 1, 2, 3$),

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \equiv Q = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}, \tag{16}$$

then

$$e_1.q = q_0 e_1 - q_1 + q_2 e_3 - q_3 e_2 \tag{17}$$

$$\equiv E_1 Q \tag{18}$$

$$= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} -q_1 \\ q_0 \\ -q_3 \\ q_2 \end{pmatrix}, \tag{19}$$

whereas[2]

$$(1|e_1).q = q \,.e_1 = q_0 e_1 - q_1 - q_2 e_3 + q_3 e_2 \tag{20}$$

$$\equiv 1|E_1 \; Q = Q E_1 \tag{21}$$

$$= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} -q_1 \\ q_0 \\ q_3 \\ -q_2 \end{pmatrix}, \tag{22}$$

and so on for the different $(e_2, e_3, 1|e_2, 1|e_3)$[3] ,

$$e_2 \equiv E_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \; , e_3 \equiv E_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{23}$$

$$1|e_2 \equiv 1|E_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \; , 1|e_3 \equiv 1|E_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{24}$$

which enable us to find any generic $e_i|e_j$

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[2]We use the elegant notations of [21].

[3] Notice that $\{E_1, E_2, E_3, 1|E_1, 1|E_2, 1|E_3\}$ are the 't Hooft matrices [22].

$$e_i|e_j. \, q = e_i \, . \, 1|e_j \, .q = e_i \, .q. \, e_j, \tag{25}$$

then we have the possible 15 combinations $\mathcal{H}|\mathcal{H}$

$$\{e_1, e_2, e_3, 1|e_1, e_1|e_1, e_2|e_1, e_3|e_1, 1|e_2, e_1|e_2, e_2|e_2, e_3|e_2, 1|e_3, e_1|e_3, e_2|e_3, e_3|e_3\}. \tag{26}$$

And their corresponding matrices

$$\{E_1, E_2, E_3, 1|E_1, E_1|E_1, E_2|E_1, E_3|E_1, 1|E_2, E_1|E_2, E_2|E_2, E_3|E_2, 1|E_3, E_1|E_3, E_2|E_3, E_3|E_3\}. \tag{27}$$

Using the matrices $\{E_1, E_2, E_3\}$, we have ($\times$ is the usual matrix multiplication),

$$E_i \times E_j = -\delta_{ij}\mathbb{1} + \epsilon_{ijk}E_k, \tag{28}$$

they satisfy the same algebra as their corresponding quaternionic units $\{e_1, e_2, e_3\}$ i.e they are isomorphic. Keep in mind this relation in order to compare it later with the octonionic case.

We can deduce the following group structure for our quaternionic operators

- Left $su(2)_L$

$$e_i.e_j = -\delta_{ij} + \epsilon_{ijk}e_k, \tag{29}$$

$$su(2)_L \sim \{e_1, e_2, e_3\}. \tag{30}$$

- Right $su(2)_R$

$$1|e_i.1|e_j = 1|(e_j.e_i) = -\delta_{ij} + \epsilon_{jik}1|e_k, \tag{31}$$

$$su(2)_R \sim \{1|e_1, 1|e_2, 1|e_3\}. \tag{32}$$

This rule can be also explicitly derived using $\{1|E_1, 1|E_2, 1|E_3\}$.

- $so(4) \sim su(2)_L \otimes su(2)_R$, which can be proved using (29) and (31) and

$$e_i.1|e_j = 1|e_j.e_i = e_i|e_j, \quad \text{i.e} \quad [e_i, \, 1|e_j] = 0, \tag{33}$$

$$so(4) \sim \{e_1, e_2, e_3, 1|e_1, 1|e_2, 1|e_3\}. \tag{34}$$

A weak form of (33), as we will see later, holds for octonions.

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- $spin(3,2)$ - and its subgroups - which can be proved by a Clifford Algebra construction

$$\gamma_0 = e_1|e_1, \quad \gamma_1 = e_3, \quad \gamma_2 = e_1|e_2, \quad \gamma_3 = e_2, \quad \gamma_5 = e_1|e_3, \tag{35}$$

$$\{\gamma_\alpha, \gamma_\beta\} = 2diag(+, -, +, -, +). \tag{36}$$

Using the matrix form (19–24) and (25), we can calculate these γ's, we find

$$\gamma_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{37}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{38}$$

$$\gamma_5 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{39}$$

It is clear that these matrices are are nothing but the famous Dirac representation up to a minus sign and the standard γ_2 is multiplied by $-i$. By explicit calculation, one finds (in the basis given above)

$$spin(3,2) \sim \{[\gamma_\alpha, \gamma_\beta]\} \qquad \alpha, \beta = 0, 1, 2, 3, 5 \quad , \tag{40}$$

$$\sim \{e_1, 1|e_1, 1|e_2, 1|e_3, e_2|e_1, e_3|e_1, e_2|e_2, e_3|e_2, e_2|e_3, e_3|e_3\}. \tag{41}$$

Actually, the main reason for this $spin(3,2)$ is the following relation

$$e_i.e_j. \ 1|e_k + e_j.e_i. \ 1|e_k = 0, \tag{42}$$

this construction is well known since a long time and used by Synge [23] to give a quaternionic formulation of special relativity ($so(1,3)$) but we don't know who was the first to derive it (most probable is Conway but the reference is too old and rare to find). Just we have done it here explicitly in parallel with their corresponding 4×4 real matrices and avoided the use of complexified quaternions.

- Also at the matrix level the full set $\mathcal{H}|\mathcal{H}$ closes an algebra,

$$1|e_i.e_j|e_k = \epsilon_{kil}e_j|e_l, \tag{43}$$

$$e_i.e_j|e_k = e_j|e_k.e_i = \epsilon_{ijl}e_l|e_k, \tag{44}$$

$$e_i|e_j.e_m|e_n = \epsilon_{iml}\epsilon_{njp}e_l|e_p. \tag{45}$$

By explicit calculations, we found that it is impossible to construct a sixth γ from the set $\mathcal{H}|\mathcal{H}$. The easiest way to see this, is to use (35) and calculate

$$\gamma_0.\gamma_1.\gamma_2.\gamma_3.\gamma_5 = -1, \tag{46}$$

so it does not form any $spin(n,m)$ higher than $spin(3,2)$.

- Adding the identity to $\mathcal{H}|\mathcal{H}$, we used Mathematica to prove that these 16 matrices are linearly independent so they can form a basis for any $R(4)$ as we claimed in (2).

A big difference between octonions and quaternions is the following : All the last equations can be reproduced by matrices **exactly** by replacing $e \longrightarrow E$ i.e there is an isomorphism between (26) and (27). The isomorphism can be derived explicitly between (28) and (29) ,then by deriving the suitable rules at the quaternionic level (31,33,43,44,45), it can be extended to the whole set of left and right actions as well as their mixing. In the octonionic case only the Clifford algebraic construction resists and holds.

IV. OCTONIONS

Moving to octonions, we use the symbols e_i to denote the imaginary octonionic units where $i, j, k = 1..7$ and

$$e_i \cdot e_j = -\delta_{ij} + \epsilon_{ijk}e_k \quad \text{or} \quad [e_i, e_j] = 2\epsilon_{ijk}e_k, \tag{47}$$

such that ϵ_{ijk} equals 1 for one of the following seven combinations $\{(123), (145), (176), (246),$ $(257), (347), (365)\}$, also, we use the symbol g to represent a generic octonionic number, and its corresponding element over $\mathcal{R}^8$ is denoted by G. As octonions are non-associative, we meet new surprises. We would like to sketch the main differences between the octonionic and the quaternionic case outlining briefly the construction given in [19,24] and focusing on the Clifford algebraic structure : The translation rules go exactly in the same manner as (16-24), just we have to use this time $R(8)$ instead of $R(4)$, we find

- First: Our left and right matrices are no more isomorphic to the octonionic algebra, for left action, we have

$$[E_i, E_j] = 2\epsilon_{ijk}E_k - 2[E_i, 1|E_j], \tag{48}$$

while

$$[e_i, e_j] = 2\epsilon_{ijk}e_k, \tag{49}$$

so the isomorphism at the level of algebra is lost and actually can never be restored as matrices are associative but octonions are not. Moreover the set $\{E_i\}$ alone does not close an algebra. Including the right action in our treatment is an obligation not a choice, then, we will be able to find something useful as we will see.

For right action, the situation is the following

$$[1|E_i, 1|E_j] = 2\epsilon_{jik}1|E_k - 2[E_i, 1|E_j], \tag{50}$$

and we have

$$[1|e_i, 1|e_j] = 2\epsilon_{jik} \ 1|e_k. \tag{51}$$

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- Second: The anticommutation relations hold at the octonionic and matrix level

$$\{e_i, e_j\} = \{1|e_i, 1|e_j\} = -2\delta_{ij}, \tag{52}$$

and the same for E_i and $1|E_i$,

$$\{E_i, E_j\} = \{1|E_i, 1|E_j\} = -2\delta_{ij}\mathbb{1}. \tag{53}$$

So a Clifford algebraic construction will be possible.

- Third: Due to the non-associativity,

$$(e_1.(e_2.g)) \neq ((e_1.e_2).g), \tag{54}$$

we have to introduce left/right octonionic operators,

$$e_i(e_j \ .g = e_i \ . \ (g.e_j) \equiv R_{ij} \times G, \tag{55}$$

$$e_i)e_j \ .g = (e_i.g) \ . \ e_j \equiv L_{ij} \times G, \tag{56}$$

which can be constructed from the following sets, $\{e_1, \ \dots \ , e_7, 1|e_1, \ \dots \ , 1|e_7\}$ and $\{E_1, \ \dots \ , E_7, 1|E_1, \ \dots \ , 1|E_7\}$, as follows

$$e_i(e_j.g = e_i.1|e_j.g \equiv R_{ij} = E_i \times 1|E_j \times G, \tag{57}$$

$$e_i)e_j.g = 1|e_j.e_i.g \equiv L_{ij} = 1|E_j \times E_i \times G. \tag{58}$$

We have given the matrix form of $\{E_1, \ \dots \ , E_7, 1|E_1, \ \dots \ , 1|E_7\}$ in a separate appendix. Having this set, we can form the different 106 elements

$$
\begin{aligned}
1, \ e_m, \ 1 \mid e_m \qquad & (15 elements), \\
e_m \mid e_m \qquad & (7), \\
e_m \ ) \ e_n \quad (m \neq n) \qquad & (42), \\
e_m \ (\ e_n \quad (m \neq n) \qquad & (42), \\
(m, \ n = 1, \dots, 7) \quad & .
\end{aligned}
\tag{59}
$$

but the two sets of 42 left/right operators are actually linearly dependent , so we should constrain ourselves to either one of them leaving for us just 64 elements. Thus, the use of this left/right operators is necessary to overcome the non-associativity problem and, at the same time, to form a basis of $R(8)$.

V. THE CLIFFORD ALGEBRAIC CONSTRUCTION

The easiest way to construct a Lie algebra from our left/right octonionic operator is to use a Clifford algebraic construction. As it is clear from (52), any of the set $\{e_i\}$ or $\{1|e_i\}$ gives an octonionic representation of Cliff(0,7) which can be represented by the matrices $\{E_i\}$ or $\{1|E_i\}$ allowing us to construct, for example, the following spin algebra :

- Matrix representation of $so(7)_L$

$$so(7) \sim \{[E_i, E_j]\} \qquad i, j = 1...7 \tag{60}$$

- Matrix representation of $so(7)_R$

$$so(7) \sim \{[1|E_i, 1|E_j]\} \qquad i, j = 1...7 \tag{61}$$

- Matrix representation of $so(8)_L$

$$so(8) \sim S_7 \otimes so(7) \sim \{E_i, [E_i, E_j]\} \qquad i, j = 1...7 \tag{62}$$

- Matrix representation of $so(8)_R$

$$so(8) \sim S_7 \otimes so(7) \sim \{1|E_i, [1|E_i, 1|E_j]\} \qquad i, j = 1...7 \tag{63}$$

where S_7 is the Reimannian seven sphere.

Notice that, from (47), we can not use the e_i as they are non-associative and consequently they don't satisfy the Jacobi identity. In summary, whatever our left/right matrices do

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not form an isomorphic representation of our left/right octonionic operators, they admit an isomorphic Clifford algebra. Now, trying to have something larger than Cliff(0,7) like the quaternionic Cliff(3,2) (eqn. 36), one would try

$$\gamma_0 \rightarrow e_2, \quad \gamma_1 \rightarrow e_3, \quad \gamma_2 \rightarrow e_4,$$

$$\gamma_3 \rightarrow e_5, \quad \gamma_4 \rightarrow e_6, \quad \gamma_5 \rightarrow e_7,$$

$$\gamma_6 \rightarrow e_1(e_1, \quad \gamma_7 \rightarrow e_1(e_2, \quad \gamma_8 \rightarrow e_1(e_3,$$

$$\gamma_9 \rightarrow e_1(e_4, \quad \gamma_{10} \rightarrow e_1(e_5, \quad \gamma_{11} \rightarrow e_1(e_6,$$

$$\gamma_{13} \rightarrow e_1(e_7. \tag{64}$$

But this construction works well for $\gamma_{0..5}$ but fails elsewhere, for example

$$\{\gamma_0, \gamma_1\}g = e_2(e_3(g_0 e_0 + g_1 e_1 + g_2 e_2 + g_3 e_3 + g_4 e_4 + g_5 e_5 + g_6 e_6 + g_7 e_7))$$

$$+ e_3(e_2(g_0 e_0 + g_1 e_1 + g_2 e_2 + g_3 e_3 + g_4 e_4 + g_5 e_5 + g_6 e_6 + g_7 e_7))$$

$$= 0. \tag{65}$$

whereas

$$\{\gamma_0, \gamma_8\} = e_2(e_1((g_0 e_0 + g_1 e_1 + g_2 e_2 + g_3 e_3 + g_4 e_4 + g_5 e_5 + g_6 e_6 + g_7 e_7)e_3))$$

$$+ e_1((e_2(g_0 e_0 + g_1 e_1 + g_2 e_2 + g_3 e_3 + g_4 e_4 + g_5 e_5 + g_6 e_6 + g_7 e_7))e_3)$$

$$\neq 0. \tag{66}$$

One may give up and say octonions are different from quaternions and they are non-associative. But, because of this reason, we still have more freedom. By a careful analysis of (66), it becomes clear that the reason of the failure is

$$E_i \times 1 | E_j \neq 1 | E_j \times E_i \tag{67}$$

a weaker form holds

$$E_i \times 1 | E_i = 1 | E_i \times E_i \tag{68}$$

in complete contrast with (33). The solution can be found to get around this problem.

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Because of the non-associativity, we should give to left and right action different priorities. As a matter of fact, this is a very reasonable requirement. When we transferred from complex numbers to quaternions, we introduced barred operators in order to overcome the non-commutativity problem and we defined their consistent rules, so going to octonions, we should expect more rules.

Assuming higher priority to right action i.e

$$e_1(e_2 \, . \, e_4 \, . \, g \equiv (e_1.(e_4.(g.e_2))), \tag{69}$$

$$e_4 \, . \, e_1(e_2 \, . \, g \equiv (e_4.(e_1.(g.e_2))). \tag{70}$$

then

$$\{ \, e_1(e_2 \, , \, e_4 \, \}.g = 0. \tag{71}$$

Using these simple rules, we can generalize (35). Using the following identities

$$\{E_i, E_j\} = -2\delta_{ij}\mathbb{1}, \tag{72}$$

$$\{1|E_i, 1|E_j\} = -2\delta_{ij}\mathbb{1}, \tag{73}$$

$$E_i \times E_j \times 1|E_k + E_j \times E_i \times 1|E_k = 0, \tag{74}$$

which hold equally well at the octonionic level

$$\{e_i, e_j\} = -2\delta_{ij}, \tag{75}$$

$$\{1|e_i, 1|e_j\} = -2\delta_{ij}, \tag{76}$$

$$e_i.e_j.1|e_k + e_j.e_i.1|e_k = 0, \tag{77}$$

in complete analogy with (42). Now, we have the possibility to write down the $Cliff(7,6)$ which is given in (64).

When any of the $\gamma_{6..13}$'s are translated into matrices, each one has two different forms, depends of being acted from right or left, e.g.

$$\gamma_0\gamma_9 = e_2.e_1(e_4.g \equiv E_2 \times E_1 \times 1|E_4 \times G, \tag{78}$$

$$\gamma_9\gamma_0 = e_1(e_4.e_2.g \equiv E_1 \times E_2 \times 1|E_4 \times G, \tag{79}$$

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they don't have a faithful 8×8 **matrix representation**. To be clear, in (78), we say that γ_9 is represented by the matrix $E_1 \times 1|E_4$ but in (79) this statement is not valid anymore as E_2 is now sandwiched between the E_1 and $1|E_4$. This is a very important fact and should be always taken into account. When we count the numbers of degrees of freedom, we have 64 for left action and 64 for right action, in total 128 real parameters which are enough to represent our $Cliff(7,6)$.

Actually, because octonions are non-associative, sometimes, we can do with them what we can not do with matrices in a straightforward way.

VI. CONCLUSION AND FURTHER DEVELOPMENTS

As it happened many times, physics may lead mathematics. We had a problem in the start. Guided by quaternions, and using octonions, we found a simple way to do $Cliff(7,6)$ by 8×8 matrices. One would, even from the start, work with matrices but we think it wouldn't be obvious at all that our γ's are represented by two different matrices depending upon left or right action.

Apart from the S_7 compactification of the D=11 supergravity and the exceptional Gut groups, the application of octonions in physics may be called "The Art of Conjectures". As they are non-associative, it is widely believed that they don't admit any Hilbert space construction. Fortunately, this is completely wrong as had been shown in [25], where many of the machinery of functional analysis had been given for this Octonionic Hilbert space by imposing different forms of scalar product (also see [26]). Mainly, such construction uses a $Cliff(0,7)$ instead of using S_7 [4]. Such techniques had been applied successfully at the quaternionic level where the complex scalar product plays a fundamental role. A complete formulation of the standard model had been given in [28] and we had extended such methods

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[4]The idea is somehow similar to the construction of the index theorems where one uses the Clifford bundle instead of the de Rham bundle of exterior differential forms [27].

to the octonionic level [29].

Finally, we want to comment about the possible further applications and investigations:

1- The Green-Schwarz string action in $D = 10$ depends on a 16-real components Majoranna-Weyl spinor, the κ symmetry removes half of these fermionic degrees of freedom leaving the action depends on just 8 real fermionic components i.e one octonion [30]. Since, there is **no way** to find D=10 dimensions Clifford algebra 8×8 gamma matrices, this represents an obstacle towards a covariant string formulation. Our representation is dependent on exactly one octonion i.e 8 real components. Actually, this was the main motive of this work. Superstring exists and without any doubt it is our best candidate for the dreamed theory of every thing, finding its true formulation is highly required. Can it be the octonionic string [13]!

2- The unified 10-13 dimensions octonionic representation is in agreement with the recent discovery of 13 hidden dimensions in string theory [31]. It would be easier to work with one octonionic construction instead of 32 components gamma matrices.

3- What is the real meaning of the different p-branes dualities? May be nothing but a non-trivial mapping between their different – postulated – infinite dimensional world-volume symmetries. Or more attractively, different mapping between different infinite dimensional ring-division superconformal algebra which may be the real connection between the ring-division algebra and the p-brane program. One of the simple formula that holds for many p-branes is

$$D - p = 2^n \qquad n = 0, 1, 2, 3. \tag{80}$$

Does it really mean that any consistent p-brane should enjoy a superconformal algebra on its transverse dimensions? This can be an amplified form of our old problem, what is the correct relation between the string sheet and the target space formulation of string theory?

We understand that the approach discussed here may not be the best in the market but with our potentially need of developing and examining the recent string dualities, it seems worthwhile to try every possible avenue.

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We introduce the following notation:

$$\{a,b,c,d\}_{(1)} \equiv \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ c & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix}, \{a,b,c,d\}_{(2)} \equiv \begin{pmatrix} 0 & a & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix}, \quad \text{(A1)}$$

$$\{a,b,c,d\}_{(3)} \equiv , \{a,b,c,d\}_{(4)} \equiv$$

where a, b, c, d and 0 represent 2×2 real matrices.

In the following σ_1, σ_2, σ_3 represent the standard Pauli matrices.

$$
\begin{aligned}
E_1 &= \{-i\sigma_2, -i\sigma_2, -i\sigma_2, i\sigma_2\}_{(1)} & , \quad 1\,|\,E_1 &= \{-i\sigma_2, i\sigma_2, i\sigma_2, -i\sigma_2\}_{(1)} , \\
E_2 &= \{-\sigma_3, \sigma_3, -1, 1\}_{(2)} & , \quad 1\,|\,E_2 &= \{-1, 1, 1, -1\}_{(2)} , \\
E_3 &= \{-\sigma_1, \sigma_1, -i\sigma_2, -i\sigma_2\}_{(2)} & , \quad 1\,|\,E_3 &= \{-i\sigma_2, -i\sigma_2, i\sigma_2, i\sigma_2\}_{(2)} , \\
E_4 &= \{-\sigma_3, 1, \sigma_3, -1\}_{(3)} & , \quad 1\,|\,E_4 &= \{-1, -1, 1, 1\}_{(3)} , \\
E_5 &= \{-\sigma_1, i\sigma_2, \sigma_1, i\sigma_2\}_{(3)} & , \quad 1\,|\,E_5 &= \{-i\sigma_2, -i\sigma_2, -i\sigma_2, -i\sigma_2\}_{(3)} , \\
E_6 &= \{-1, -\sigma_3, \sigma_3, 1\}_{(4)} & , \quad 1\,|\,E_6 &= \{-\sigma_3, \sigma_3, -\sigma_3, \sigma_3\}_{(4)} , \\
E_7 &= \{-i\sigma_2, -\sigma_1, \sigma_1, -i\sigma_2\}_{(4)} & , \quad 1\,|\,E_7 &= \{-\sigma_1, \sigma_1, -\sigma_1, \sigma_1\}_{(4)} .
\end{aligned}
\tag{A2}
$$

By explicit calculation, [32], one finds that these matrices are the direct generalization of the 't Hooft matrices [22], for $a = 1..7$,

$$(E_a)_{\mu\nu} = \epsilon_{a\mu\nu} \qquad\qquad \mu, \nu = 1..7 \tag{A3}$$

$$(1|E_a)_{\mu\nu} = -\epsilon_{a\mu\nu} \qquad\qquad \mu, \nu = 1..7 \tag{A4}$$

$$(E_a)_{0\nu} = (1|E_a)_{0\nu} = -\delta_{a\nu} \qquad \mu, \nu = 0..7 \tag{A5}$$

$$(E_a)_{\mu 0} = (1|E_a)_{\mu 0} = \delta_{a\mu} \qquad \mu, \nu = 0..7 \tag{A6}$$

$$(E_a)_{00} = (1|E_a)_{00} = 0. \tag{A7}$$